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18 October 2001

PHYSICS LETTERS B

Physics Letters B 518 (2001) 301–305

www.elsevier.com/locate/npe

A note on conifolds

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Received 27 July 2001; accepted 20 August 2001

Editor: T. Yanagida

Abstract

We present the Ricci-flat metric and its Kähler potential on the conifold with the $O(N)$ isometry, whose conical singularity is repaired by the complex quadric surface $Q^{N-2} = SO(N)/SO(N-2) \times U(1)$.

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1. Introduction

Conformally invariant nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in two-dimensions describe the superstring in curved space. The target space must be a Ricci-flat Kähler manifold by the requirement of finiteness [1–3]. In the previous letter [4], we presented the simple derivation of the Ricci-flat metric on the deformed conifold with the $O(N)$ isometry, whose conical singularity is removed by S^{N-1} . It coincides with the Stenzel metric on the cotangent bundle over S^{N-1} [5], and includes the Eguchi–Hanson gravitational instanton [6] and the six-dimensional deformed conifold [7,8] in the cases of $N = 3$ and $N = 4$, respectively. The metric contains the deformation parameter, and the manifold becomes a conifold when the parameter vanishes.

In this Letter, we present the explicit form of the Ricci-flat metric and its Kähler potential on the conifold, whose conical singularity is repaired by the complex quadric surface $Q^{N-2} \equiv SO(N)/SO(N-2) \times U(1)$. It contains a resolution parameter b as an in-

tegration constant, which controls the size of Q^{N-2} . The limit of $b \rightarrow 0$ corresponds to the conifold, which coincides with the singular limit of the deformed conifold. Our manifold can be interpreted as the line bundle over Q^{N-2} . The four-dimensional manifold of $N = 3$ is again the Eguchi–Hanson space, in which the conical singularity is removed by $Q^1 \simeq S^2$. In the case of the six-dimensional manifold of $N = 4$, the conical singularity is repaired by $Q^2 \simeq S^2 \times S^2$, and it gives a way to repair the singularity different from the deformation by S^3 [7,8] or the so-called *small resolution* by S^2 [7,9].

2. Definition of the model

$\mathcal{N} = 2$ supersymmetric nonlinear sigma models in two dimensions are described by the chiral superfields $\varphi^\alpha(x, \theta, \bar{\theta})$ and the Kähler potential $\mathcal{K}(\varphi, \varphi^*)$ [10]. The Lagrangian is given by $\mathcal{L} = \int d^4\theta \mathcal{K} = g_{\alpha\beta^*} \times (\varphi, \varphi^*) \partial_\mu \varphi^\alpha \partial^\mu \varphi^{*\beta} + \dots$, where the Kähler metric is defined by $g_{\alpha\beta^*} = \partial_\alpha \partial_{\beta^*} \mathcal{K}$ with $\partial_\alpha = \partial/\partial\varphi^\alpha$ and $\partial_{\alpha^*} = \partial/\partial\varphi^{*\alpha}$. (Here we have used the same letters for chiral superfields and their components.)

First, we prepare chiral superfields $\phi^A(x, \theta, \bar{\theta})$ ($A = 1, 2, \dots, N$; $N \geq 3$), constituting the vector

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$\vec{\phi}(x, \theta, \bar{\theta})$ of $O(N)$. We define the $O(N)$ symmetric target space by imposing the constraint

$$\sum_{A=1}^N (\phi^A)^2 = 0. \quad (1)$$

This constraint defines the conifold with the real dimension $2N - 2$. We can rewrite this by an unitary transformation as

$$\vec{\phi}^T J \vec{\phi} = 0. \quad (2)$$

Here J is the rank-2 invariant tensor of $O(N)$, which we take as

$$J = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad (3)$$

where $\mathbf{1}_{N-2}$ is the $(N - 2) \times (N - 2)$ unit matrix.

Introducing an auxiliary chiral superfield $\phi_0(x, \theta, \bar{\theta})$, we can give the $O(N)$ symmetric Lagrangian by

$$\mathcal{L} = \int d^4\theta \mathcal{K}(X) + \left(\int d^2\theta \phi_0 \vec{\phi}^T J \vec{\phi} + \text{c.c.} \right). \quad (4)$$

Here, $X(x, \theta, \bar{\theta})$ is the $O(N)$ -invariant real superfield, defined by

$$X \equiv \sum_{A=1}^N \phi^{\dagger A} \phi^A, \quad (5)$$

and $\mathcal{K}(X)$ is an arbitrary function of X . The symmetry of the Lagrangian (4) is $G = O(N) \times U(1)$, assigning the $U(1)$ charges of ϕ^A and ϕ_0 , 1 and -2 , respectively. By the integration over the auxiliary field ϕ_0 , we obtain the constraint (2), which can be immediately solved as

$$\vec{\phi} = \sigma \begin{pmatrix} 1 \\ z^i \\ -\frac{1}{2}(z^i)^2 \end{pmatrix}, \quad (6)$$

where the summation over the repeated indices is implied. Here $\sigma(x, \theta, \bar{\theta})$ and $z^i(x, \theta, \bar{\theta})$ ($i = 1, 2, \dots, N - 2$) are chiral superfields, with the $U(1)$ charges 1 and 0, respectively. Scalar components of these superfields parameterize the target space, and the symmetry G acts on those fields as a holomorphic isometry. The invariant X becomes

$$X = |\sigma|^2 \left[1 + |z^i|^2 + \frac{1}{4}(z^i)^2 (z^{*j})^2 \right] \equiv |\sigma|^2 Z. \quad (7)$$

Note that the constraint (1) or (2) is invariant under the complex extension of the symmetry G . Using this, any point $\vec{\phi}$ on the manifold can be transformed to $\langle \vec{\phi}^T \rangle = (1, 0, \dots, 0)$, which can be interpreted as the vacuum expectation value. From this, we find the symmetry G is spontaneously broken down to $H = O(N - 2) \times U(1)$. Hence there appear the Nambu–Goldstone bosons, parameterizing $G/H \simeq SO(N)/SO(N - 2)$. The whole target manifold can be locally regarded as $\mathbf{R} \times SO(N)/SO(N - 2) \simeq \mathbf{R} \times S^{N-1} \times S^{N-2}$.

3. Ricci-flat condition and its solution

We would like to determine the function $\mathcal{K}(X)$, imposing the Ricci-flat condition on the manifold. We use the *same* letters for superfields *and* their lowest components from now on. The Kähler metric is

$$g_{\alpha\beta^*}(\varphi, \varphi^*) = \frac{\partial^2 \mathcal{K}(X)}{\partial \varphi^\alpha \partial \varphi^{*\beta}} = \frac{d^2 \mathcal{K}}{dX^2} \frac{\partial X}{\partial \varphi^\alpha} \frac{\partial X}{\partial \varphi^{*\beta}} + \frac{d\mathcal{K}}{dX} \frac{\partial^2 X}{\partial \varphi^\alpha \partial \varphi^{*\beta}}, \quad (8)$$

where $\varphi^\alpha \equiv (\sigma, z^i)$. The Ricci form is given by $(\text{Ric})_{\alpha\beta^*} = -\partial_\alpha \partial_{\beta^*} \log \det g_{\gamma\delta^*}$, and the Ricci-flat condition $(\text{Ric})_{\alpha\beta^*} = 0$ implies $\det g_{\alpha\beta^*} = (\text{constant}) \times |F|^2$, with F being a holomorphic function. This is a partial differential equation, which is difficult to solve in general. The determinant $\det g_{\alpha\beta^*}$ can be calculated as

$$\det g_{\alpha\beta^*} = \frac{X}{|\sigma|^2} \left(X \frac{d^2 \mathcal{K}}{dX^2} + \frac{d\mathcal{K}}{dX} \right) \left(|\sigma|^2 \frac{d\mathcal{K}}{dX} \right)^{N-2} \times \det(\partial_i \partial_{j^*} Z - Z^{-1} \partial_i Z \partial_{j^*} Z), \quad (9)$$

where ∂_i denotes the differentiation with respect to z^i : $\partial_i Z = z^{*i} + \frac{1}{2} z^i (z^{*j})^2$ and $\partial_i \partial_{j^*} Z = \delta_{ij} + z^i z^{*j}$. Using the complex extension of the isotropy H , $SO(N - 2, \mathbf{C})$, we can choose a point labeled by $z^1 \neq 0$ and $z^m = 0$ ($m = 2, 3, \dots, N - 2$), without loss of generality. At that point, we find

$$\det(\partial_i \partial_{j^*} Z - Z^{-1} \partial_i Z \partial_{j^*} Z) = \det \delta_{ij} = 1, \quad X = |\sigma|^2 \left(1 + \frac{|z^1|^2}{2} \right)^2, \quad (10)$$

and then obtain

$$\det g_{\alpha\beta^*} = |\sigma|^{2N-6} \left(\frac{d\mathcal{K}}{dX} \right)^{N-2} \left(X^2 \frac{d^2\mathcal{K}}{dX^2} + X \frac{d\mathcal{K}}{dX} \right). \quad (11)$$

Therefore, the Ricci-flat condition becomes an ordinary differential equation:

$$\begin{aligned} & \left(\frac{d\mathcal{K}}{dX} \right)^{N-2} \left(X^2 \frac{d^2\mathcal{K}}{dX^2} + X \frac{d\mathcal{K}}{dX} \right) \\ &= \frac{1}{N-1} X^2 \frac{d}{dX} \left[\left(\frac{d\mathcal{K}}{dX} \right)^{N-1} \right] + X \left(\frac{d\mathcal{K}}{dX} \right)^{N-1} \\ &\equiv c, \end{aligned} \quad (12)$$

where c is a constant. This can be immediately solved as

$$\frac{d\mathcal{K}}{dX} = \frac{(\lambda X^{N-2} + b)^{\frac{1}{N-1}}}{X}, \quad (13)$$

where λ is a constant related to c and N , and b is an integration constant. We impose $b \geq 0$ and $\lambda > 0$ in order that the Kähler potential is real.

The solution (13) is sufficient to obtain the Ricci-flat metric using (8), but we can calculate its Kähler potential by integrating (13):

$$\begin{aligned} \mathcal{K}(X) &= \frac{N-1}{N-2} \left[(\lambda X^{N-2} + b)^{\frac{1}{N-1}} \right. \\ &\quad \left. + b^{\frac{1}{N-1}} \cdot I \left(b^{\frac{1}{1-N}} (\lambda X^{N-2} + b)^{\frac{1}{N-1}}; N-1 \right) \right], \end{aligned} \quad (14)$$

where the function $I(y; n = N-1)$ is defined by

$$\begin{aligned} I(y; n) &\equiv \int_1^y \frac{dt}{t^n - 1} \\ &= \frac{1}{n} \left[\log(y-1) - \frac{1+(-1)^n}{2} \log(y+1) \right] \\ &\quad + \frac{1}{n} \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos \frac{2r\pi}{n} \log \left(y^2 - 2y \cos \frac{2r\pi}{n} + 1 \right) \\ &\quad + \frac{2}{n} \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \sin \frac{2r\pi}{n} \arctan \left[\frac{\cos(2r\pi/n) - y}{\sin(2r\pi/n)} \right]. \end{aligned} \quad (15)$$

If we set $b = 0$ in (14), it becomes the Kähler potential of the conifold, which coincides with the one of the singular limit of the deformed conifold [4].

4. Ricci-flat metric

Using Eqs. (8) and (13), the components of the Ricci-flat metric can be calculated, to give

$$g_{\sigma\sigma^*} = \lambda \left(\frac{N-2}{N-1} \right) (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} X^{N-2} |\sigma|^{-2}, \quad (16a)$$

$$g_{\sigma j^*} = \lambda \left(\frac{N-2}{N-1} \right) (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} X^{N-3} \sigma^* \partial_{j^*} Z, \quad (16b)$$

$$\begin{aligned} g_{ij^*} &= \lambda \left(\frac{N-2}{N-1} \right) (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} \\ &\quad \times X^{N-4} |\sigma|^4 \partial_i Z \partial_{j^*} Z \\ &\quad + (\lambda X^{N-2} + b)^{\frac{1}{N-1}} \\ &\quad \times (Z^{-1} \partial_i \partial_{j^*} Z - Z^{-2} \partial_i Z \partial_{j^*} Z). \end{aligned} \quad (16c)$$

This Kähler metric is singular at the surface defined by $\sigma = 0$: $g_{\sigma\sigma^*}|_{\sigma=0} = 0$. However this is just a coordinate singularity of the coordinate system (σ, z^i) . To find regular coordinates, let us perform a coordinate transformation

$$\rho \equiv \frac{\sigma^{N-2}}{N-2}, \quad (17)$$

with z^i being unchanged. The components of the Kähler metric in the new coordinates (ρ, z^i) are

$$g_{\rho\rho^*} = \lambda \left(\frac{N-2}{N-1} \right) (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} Z^{N-2}, \quad (18a)$$

$$\begin{aligned} g_{\rho j^*} &= \lambda \frac{(N-2)^2}{N-1} (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} \\ &\quad \times \rho^* Z^{N-3} \partial_{j^*} Z, \end{aligned} \quad (18b)$$

$$\begin{aligned} g_{ij^*} &= \lambda \frac{(N-2)^3}{N-1} (\lambda X^{N-2} + b)^{\frac{2-N}{N-1}} \\ &\quad \times |\rho|^2 Z^{N-4} \partial_i Z \partial_{j^*} Z \\ &\quad + (\lambda X^{N-2} + b)^{\frac{1}{N-1}} \\ &\quad \times (Z^{-1} \partial_i \partial_{j^*} Z - Z^{-2} \partial_i Z \partial_{j^*} Z), \end{aligned} \quad (18c)$$

where $X = |(N-2)\rho|^{\frac{2}{N-2}} Z$. These are non-singular at the surface of $\rho = 0$, corresponding to $\sigma = 0$, as long as the integration constant b takes a non-zero value. In the limit of $b \rightarrow 0$, the manifold becomes the conifold and the metric (18) becomes singular at $\rho = 0$. So we can regard this constant b as a *resolution* parameter of

the conical singularity. The coordinate singularity in the coordinates (σ, z^i) is due to the identification of (17) as in the Calabi metric on the line bundle over $\mathbb{C}P^{N-1}$ [11].

The metric of the $\rho = 0$ surface itself ($d\rho = 0$) is

$$g_{ij^*}(z, z^*) = b^{\frac{1}{N-1}} (Z^{-1} \partial_i \partial_{j^*} Z - Z^{-2} \partial_i Z \partial_{j^*} Z). \quad (19)$$

This defines a Kähler submanifold whose Kähler potential is given by

$$\begin{aligned} \mathcal{K}(z, z^*) &= b^{\frac{1}{N-1}} \log \left[1 + |z^i|^2 + \frac{1}{4} (z^i)^2 (z^{*j})^2 \right] \\ &= b^{\frac{1}{N-1}} \log Z, \end{aligned} \quad (20)$$

which is the Kähler potential of the complex quadric surface $Q^{N-2} = SO(N)/SO(N-2) \times U(1)$ [12–15]. Therefore we have found that the conical singularity is resolved by Q^{N-2} of the radius $b^{\frac{1}{2(N-1)}}$. The manifold can be interpreted as the line bundle over Q^{N-2} . In fact it was proved in [16] that there exists a Ricci-flat Kähler metric on the line bundle over any Einstein manifold.

5. Examples

Let us give the more concrete expressions for the $N = 3$ and $N = 4$ cases. For the four-dimensional manifold of $N = 3$, the Kähler potential (14) becomes

$$\mathcal{K}(X) = 2\sqrt{\lambda X + b} + \sqrt{b} \log \left(\frac{\sqrt{\lambda X + b} - \sqrt{b}}{\sqrt{\lambda X + b} + \sqrt{b}} \right). \quad (21)$$

Defining $\varrho^4 = 4(\lambda X + b)$ and $a^4 = 4b$, we find that this is the Kähler potential [17] of the Eguchi–Hanson gravitational instanton [6]:

$$\mathcal{K} = \varrho^2 + \frac{a^2}{2} \log \left(\frac{\varrho^2 - a^2}{\varrho^2 + a^2} \right). \quad (22)$$

The singularity at the apex of the conifold is repaired by $Q^1 \simeq S^2$, and the isometry is $SO(3) \times U(1) \simeq U(2)$.

The Kähler potential (14) in the six-dimensional manifold of $N = 4$ is

$$\mathcal{K}(X) = \frac{3}{2} (\lambda X^2 + b)^{1/3}$$

$$\begin{aligned} &+ \frac{b^{1/3}}{4} \log \left[\frac{(\lambda X^2 + b)^{1/3} - b^{1/3}}{\lambda X^2} \right] \\ &- \frac{\sqrt{3} b^{1/3}}{2} \arctan \left[\frac{2(\lambda X^2 + b)^{1/3} + b^{1/3}}{\sqrt{3} b^{1/3}} \right]. \end{aligned} \quad (23)$$

The metric in the coordinates (ρ, z^1, z^2) is represented as follows:

$$\begin{aligned} g_{\rho\rho^*} &= \frac{2\lambda}{3} \frac{Z^2}{(\lambda X^2 + b)^{2/3}}, \\ g_{\rho j^*} &= \frac{4\lambda}{3} \frac{\rho^* Z \partial_{j^*} Z}{(\lambda X^2 + b)^{2/3}}, \end{aligned} \quad (24a)$$

$$\begin{aligned} g_{ij^*} &= \frac{8\lambda}{3} \frac{|\rho|^2 \partial_i Z \partial_{j^*} Z}{(\lambda X^2 + b)^{2/3}} \\ &+ (\lambda X^2 + b)^{1/3} (Z^{-1} \partial_i \partial_{j^*} Z - Z^{-2} \partial_i Z \partial_{j^*} Z). \end{aligned} \quad (24b)$$

The isometry of this manifold is $SO(4) \times U(1) \simeq SU(2) \times SU(2) \times U(1)$. The singularity at the apex of the conifold is repaired by $Q^2 \simeq S^2 \times S^2$ (the radii of these two S^2 coincide). This way of repairing of the conical singularity is different from either the *deformation* by S^3 [7,8] or the *small resolution* by S^2 [7,9] known in the six-dimensional conifold.

6. Discussions

We can obtain the Kähler potential (20) of $Q^{N-2} = SO(N)/SO(N-2) \times U(1)$ directly, by gauging the $U(1)$ part of the isometry G and performing the integration over gauge superfields. (This is known as Kähler quotient [18], and actually holds for an arbitrary Kähler potential $\mathcal{K}(X)$ [13].) Replacing the base manifold Q^{N-2} by other compact manifolds in [12,14], we can construct other Ricci-flat Kähler manifolds, whose conical singularity is repaired by those base manifolds [19]. Since non-perturbative effects of Q^N was investigated using the large- N method in [15], the large- N limit of the conifold is also interesting. The investigation of super-conformal field theories corresponding to our manifolds is an interesting task.

After the completion of this work we were informed that the six-dimensional manifold in the $N = 4$ case is known in Refs. [20,21].

Acknowledgements

We would like to thank Michihiro Naka, Kazutoshi Ohta and Takashi Yokono for valuable comments and discussions. This work was supported in part by the Grant-in-Aid for Scientific Research (#13640283).

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